Numerical Analysis of Flow and Heat Transfer of a Viscoelastic Fluid Over A Stretching Sheet by Using the Homotopy Analysis Method

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Abstract

Purpose - In this paper a study of the flow and heat transfer of an incompressible homogeneous second grade fluid past a stretching sheet channel is presented and the Homotopy Analysis Method (HAM) is employed to compute an approximation to the solution of the system of nonlinear differential equations governing on the problem. It has been attempted to show the capabilities and wide-range applications of the Homotopy Analysis Method in comparison with the numerical method in solving this problems. The obtained solutions, in comparison with the exact solutions admit a remarkable accuracy. A clear conclusion can be drawn from the numerical method results that the HAM provides highly accurate solutions for nonlinear differential equations.
Design/methodology/approach - In this paper a study of the flow and heat transfer of an incompressible homogeneous second grade fluid past a stretching sheet channel is presented and the Homotopy Analysis Method (HAM) is employed to compute an approximation to the solution of the system of nonlinear differential equations governing on the problem. It has been attempted to show the capabilities and wide-range applications of the Homotopy Analysis Method in comparison with the numerical method in solving this problems. The obtained solutions, in comparison with the exact solutions admit a remarkable accuracy.
Findings - In this paper a study of the flow and heat transfer of an incompressible homogeneous second grade fluid past a stretching sheet channel is presented and the Homotopy Analysis Method (HAM) is employed to compute an approximation to the solution of the system of nonlinear differential equations governing on the problem. It has been attempted to show the capabilities and wide-range applications of the Homotopy Analysis Method in comparison with the numerical method in solving this problems. The obtained solutions, in comparison with the exact solutions admit a remarkable accuracy.
Originality/value - In this paper a study of the flow and heat transfer of an incompressible homogeneous second grade fluid past a stretching sheet channel is presented and the Homotopy Analysis Method (HAM) is employed to compute an approximation to the solution of the system of nonlinear differential equations governing on the problem. It has been attempted to show the capabilities and wide-range applications of the Homotopy Analysis Method in comparison with the numerical method in solving this problems. The obtained solutions, in comparison with the exact solutions admit a remarkable accuracy.

Keywords: Homotopy analysis method (HAM); Flow and heat transfer; Second grade fluid; Stretching sheet

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1. Introduction

The study of the flow field due to a stretching sheet in an ambient fluid is important in several practical engineering applications. Extrusion processes, fibers spinning, hot rolling, manufacturing of plastic and rubber sheet, continuous casting and glass blowing are examples of industrial applications of stretching of a surface in an ambient fluid. Since the pioneering work of Sakiadis [1], various aspects of the problem have been investigated by many authors. Good lists of references on this problem can be found in Sadeghy and Sharifi [2], and Hayat et al. [3-5]. Crane [6] and Gupta and Gupta [7] have analysed the stretching problem with constant surface temperature (CST) while Soundalgekar [8] investigated the Stokes problem for a viscoelastic fluid. This flow was examined by Siddappa and Khapate [9] for a special class of non-Newtonian fluids known as second-order fluids which are viscoelastic in nature. Danberg and Fansler [10] studied the solution for the boundary layer flow past a wall that is stretched with a speed proportional to the distance along the wall. These scientific problems and phenomena are modeled by ordinary or partial differential equations. In most cases, these problems do not admit analytical solution, so these equations should be solved using special techniques. In recent years, much attention has been devoted to the newly developed methods to construct an analytic solution of equation; such methods include the Adomian decomposition method [11], Artificial Parameter Lindstedt–Poincaré [12], Multiple Scale [13], Newton–harmonic balancing [14], differential transformation [15] and Perturbation techniques [16].

Perturbation techniques are too strongly dependent upon the so-called ''small parameters'' [16]. Thus, it is worthwhile developing some new analytic techniques independent upon small parameters. Homotopy Analysis Method (HAM), which was expected by Liao [17–22], has been applied to solve many types of nonlinear problems successfully [23-35]. In this Letter, the basic idea of the HAM is introduced and then the equations of flow and heat transfer of a viscoelastic fluid over a stretching sheet are solved through HAM. The results are compared with the exact solution of this problem which is examined by Rafael [36].

Nomenclature

A	area	s	Wall temperature parameter
c_p	Specific heat	T_∞	Ambient temperature
Ec	Eckert number	u	Velocity component in x-direction
$\hbar$	Auxiliary parameter	v	Velocity component in y-direction
HAM	Homotopy Analysis Method	*Greek symbols*	
k	Viscoelastic parameter	α_1	Thermal viscosity
$\mathcal{L}$	Linear operator of HAM	ρ	Density of the fluid
N	Non-linear operator	σ	Prandtl
NM	Numerical solution	μ	Viscosity of convective fluid
		υ	Kinematic viscosity
p	Embedding parameter		

2. Formulation of the problem

2.1. Flow analysis

We consider the flow of an incompressible second grade fluid past a flat sheet coinciding with the plane $y = 0$, the flow being confined to $y > 0$. Two equal and opposite forces are applied along the x-axis so that the wall is stretched keeping the origin fixed. The steady two-dimensional boundary layer equations for this fluid, in the usual notation, are [36]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial^2 y}+\frac{\alpha_1}{\rho}\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}\right], \tag{2}$$

where u and v are the velocity components in x and y directions, respectively, υ is the kinematic viscosity and ρ is the density. The boundary conditions to the problem are

$$u=cx, \quad v=0 \quad at \quad y=0, \quad c>0, \tag{3}$$

$$u\to 0, \quad \frac{\partial u}{\partial y}\to 0 \quad as \quad y\to\infty. \tag{4}$$

The second condition (4) is the augmented condition since the flow is in an unbounded domain, which has been discussed by Garg and Rajagopal [37].

Defining new variables

$$u=cxf'(\eta), \quad v=-(c\upsilon)^{1/2} f(\eta), \tag{5}$$

where

$$\eta=\left(\frac{c}{\upsilon}\right)^{1/2} y, \tag{6}$$

and substituting in (2) gives

$$(f')^2-ff''=f'''+k\left[2ff''-(f'')^2-ff^{iv}\right], \tag{7}$$

where $k=\frac{\alpha_1 c}{\rho\nu}$ is the viscoelastic parameter and a prime denotes differentiation with respect to η. The boundary conditions (3) and (4) become

$$f=0, \quad f'=1 \quad at \quad \eta=0, \tag{8}$$

$$f'\to 0, \quad f''\to 0 \quad at \quad \eta\to\infty. \tag{9}$$

2.2. Heat transfer analysis

By using boundary layer approximations, the equation of energy with viscous dissipation for temperature T is given by

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\upsilon}{c_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{10}$$

where α is the thermal diffusivity and c_p is the specific heat of a fluid at constant pressure and we have assumed that the radiation is negligible.

The boundary conditions are

$$T=T_w(=T_\infty+Ax^s) \quad at \quad y=0, \quad T\to T_\infty \quad as \quad y\to\infty. \tag{11}$$

where s is the wall temperature parameter.

Defining the non-dimensional temperature $\theta(\eta)$ and the Prandtl number σ as

$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}, \qquad \sigma=\frac{\upsilon}{\alpha} \tag{12}$$

Using Eqs. (5), (6) and (12), and Eq. (10) and conditions (11) can be written as

$$\theta''+\sigma f\theta'-s\sigma f'\theta=-\sigma Ec(f'')^2 x^{2-s}. \tag{13}$$

$$\theta(0)=1, \quad \theta(\infty)\to 0, \tag{14}$$

with $Ec=c^2/Ac_p$.

If $s = 2$, we find from (13)

$$\theta'' + \sigma f \theta' - 2\sigma f' \theta = -\sigma Ec (f'')^2. \tag{15}$$

It is clear from Eq. (15) that all solutions are then of the similar type. When the effect of dissipative heat is neglected, we obtain from (15) the simpler equation

$$\theta'' + \sigma f \theta' - 2\sigma f' \theta = 0. \tag{16}$$

On the other hand, for negligible dissipation, we find from Eq. (13)

$$\theta'' + \sigma f \theta' - s\sigma f' \theta = 0. \tag{17}$$

where s is now arbitrary.

3. HAM solutions

In order to obtain HAM solutions of Eqs. (7) and (16) we choose the following initial guesses and auxiliary linear operator:

$$f_0(\eta) = 1 - \exp(-\eta), \qquad \theta_0(\eta) = \exp(-\eta), \tag{18}$$

$$L_1(f) = f''' - f' \quad , L_2(\theta) = \theta'' - \theta \tag{19}$$

$$L_1(c_1 + c_2 \exp(\eta) + c_3 \exp(-\eta)) = 0, \qquad L_2(c_4 \exp(-\eta) + c_5 \exp(\eta)) = 0, \tag{20}$$

where $c_i (i = 1-5)$ are constants Let $P \in [0,1]$ denotes the embedding parameter and $\hbar_1, \hbar_2$ indicate none –zero auxiliary parameters. We then construct the following problems:

Zeroth –order deformation problems

$$(1-p)L_1[f(\eta;p) - f_0(\eta)] = p\hbar_1 N_1[f(\eta;p)] \tag{21}$$

$$f(0;p) = 0; \qquad f'(0;p) = 0; \qquad f'(\infty;p) = 0; \qquad \theta(0;p) = 1; \qquad \theta(\infty;p) = 0. \tag{22}$$

$$(1-p)L_2[\theta(\eta;p) - \theta_0(\eta)] = p\hbar_2 N_2[\theta(\eta;p), f(\eta;p)] \tag{23}$$

$$f(0;p) = 1; \qquad \theta(1;p) = 0. \tag{24}$$

$$\begin{aligned} N_1[f(\eta;p),\theta(\eta;p)] = {} & \frac{\partial^3 f(\eta;p)}{\partial\eta^3} - \left(\frac{\partial f(\eta;p)}{\partial\eta}\right)^2 - f(\eta;p)\frac{\partial^2 f(\eta;p)}{\partial^2\eta} + \\ & k\left[2f(\eta;p)\frac{\partial^3 f(\eta;p)}{\partial^3\eta} - \left(\frac{\partial^2 f(\eta;p)}{\partial^2\eta}\right)^2 - f(\eta;p)\frac{\partial^4 f(\eta;p)}{\partial^4\eta}\right] \end{aligned} \tag{25}$$

$$N_2[f(\eta;p),\theta(\eta;p)] = \frac{\partial^2\theta(\eta;p)}{\partial\eta^2} + \sigma f(\eta;p)\frac{\partial\theta(\eta;p)}{\partial\eta} - 2\sigma\theta(\eta;p)\frac{\partial f(\eta;p)}{\partial\eta} \tag{26}$$

For $p = 0$ and $p = 1$ we have

$$f(\eta;0) = f_0(\eta) \qquad f(\eta;1) = f(\eta) \tag{27}$$

$$\theta(\eta;0) = \theta_0(\eta) \qquad \theta(\eta;1) = \theta(\eta) \tag{28}$$

When p increases from *0* to *1* then $f(\eta;p)$ and $\theta(\eta;p)$ vary from $f_0(\eta)$ and $\theta_0(\eta)$ to $f(\eta)$ and $\theta(\eta)$. Due to Taylor series with respect to *p,* we have

$$f(\eta;p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \; f_m(\eta) = \frac{1}{m!}\frac{\partial^m(f(\eta;p))}{\partial p^m} \tag{29}$$

$$\theta(\eta;p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \; \theta_m(\eta) = \frac{1}{m!}\frac{\partial^m(\theta(\eta;p))}{\partial p^m} \tag{30}$$

In which $\hbar_1$ and $\hbar_2$ are chosen in such a way that these two series are convergent at $p = 1$, therefore we have through Eqs. (29, 30) that

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \ \ \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \tag{31}$$

mth –order deformation problems

$$L_1\left[f_m(\eta) - \chi_m f_{m-1}(\eta)\right] = \hbar_1 R_m^f(\eta) \tag{32}$$

$$\begin{aligned} &f(0;p) = 0; \quad f'(0;p) = 0; \quad f'(\infty;p) = 0; \quad \theta(0;p) = 1; \quad \theta(\infty;p) = 0. \\ &f_m(0) = 0; \quad f_m'(0) = 0, \quad f_m'(1) = 0; \quad f_m'(1) = 0. \end{aligned} \tag{33}$$

$$L_2\left[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)\right] = \hbar_2 R_m^\theta(\eta) \tag{34}$$

$$\theta_m(0) = 1, \qquad \theta_m(\infty) = 0. \tag{35}$$

$$R_{m+1}^f(\eta) = f_m''' - \sum_{z=0}^{m}\left(f_{m-z} f_z'' - f_{m-z}' f_z' + k\left[\sum_{z=0}^{m} f_{m-z}'' f_z'' + 2 f_{m-z}' f_z''' - f_{m-z} f_z^{IV}\right]\right) \tag{36}$$

$$R_m^\theta(\eta) = \theta_m'' + \sigma \sum_{z=0}^{m}\left(f_{m-z}\theta_z' - 2\theta_{m-z} f_{m-z}'\right) \tag{37}$$

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases} \tag{38}$$

We have found the answer by maple analytic solution device. For first deformation of the coupled solution are presented below.

$$f_1(\eta) = 1.03057 \times 10^{-8}\, \hbar_1\, k\, e^{\eta} + 0.5\hbar_1\, k\, e^{-\eta} + 0.5\hbar_1\, k\, \eta e^{-\eta} - 0.5\hbar_1\, k, \tag{39}$$

$$\begin{aligned} \theta_1(\eta) = {} & 1.03 \times 10^{-8}\, \hbar_2\, e^{-\eta} + 0.33333\hbar_2\, \sigma\, e^{-\eta} + 1.03305 \times 10^{-8}\, \hbar_2\, e^{\eta} - \\ & 1.09927 \times 10^{-8}\, \hbar_2\, \sigma\, e^{\eta} - 0.5\hbar_2\, \eta\, e^{-\eta} + 0.5\hbar_2\, \sigma\, \eta\, e^{-\eta} - 0.33333\hbar_2\, \sigma\, e^{-2\eta}, \end{aligned} \tag{40}$$

The solutions $f_2(\eta)$ and $\theta_2(\eta)$ were too long to be mentioned here, therefore, they are shown graphically. But it is necessary to remind that both auxiliary parameters of $\hbar_1$ and $\hbar_2$ appear in other terms of $\theta(\eta)$.

4. Convergence of the HAM solution

As pointed out by Liao, the convergence and rate of approximation for the HAM solution strongly depends on the values of auxiliary parameters $\hbar_1$ and $\hbar_2$. For this purpose, the $\hbar$-curves are plotted for f and θ. According to Fig. 1 the ranges for values of $\hbar_1$ is $-2 < \hbar_1 < -0.1$. Our calculations depict that the series of the velocity field in Eq. (31) converges in the whole region of η for $\hbar_1 = -1$. It is obvious from Fig. 2 that the range for the admissible value for $\hbar_2$ is $-1.25 < \hbar_2 < -0.75$. The series of heat flux in Eq. (31) converges in the whole region of η when $\hbar_2 = -1$.

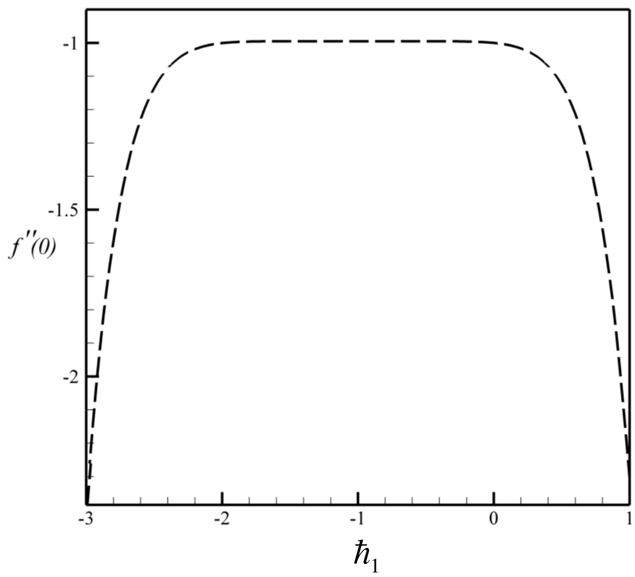

Fig. 1. The $\hbar_1$ - validity for 8th-order of approximation

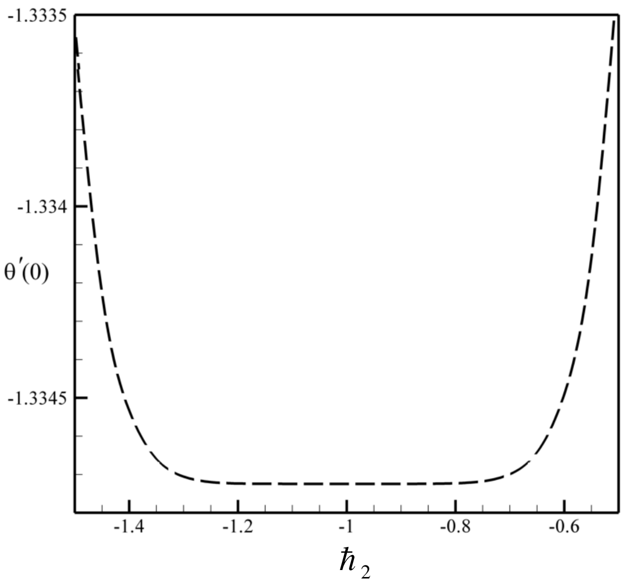

Fig. 2. The $\hbar_2$ - validity for 8th-order of approximation

5. Results and discussion

Now we discuss the variation of horizontal and vertical velocity components with distance from the surface at $\hbar_1 = -1$ for 8th order of approximations for different values of elastic parameter k.

Fig. 3 shows the variation of horizontal velocity component $f'(\eta)$ with distance from the surface η for several values of the elastic parameter k. It can be seen that the velocity increases with an increase in elasticity k of the fluid. Fig. 4 shows the variation of vertical component of velocity $f(\eta)$ with distance η from surface. It is found that when the value of k increases then the vertical velocity at a point increases.

Graphs are also plotted for the temperature distribution $\theta(\eta)$ for 8th order of approximations. The temperature profiles $\theta(\eta)$ in the absence of heat dissipation are plotted against η.

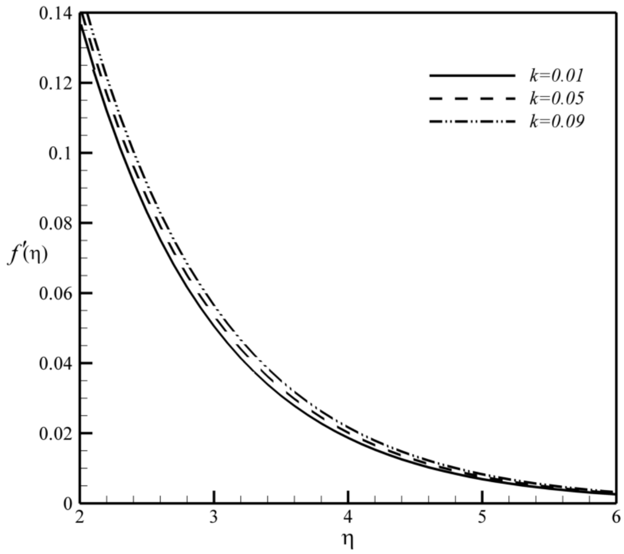

Fig. 3. horizontal velocity profiles for several values of *k* with $\sigma = 1$

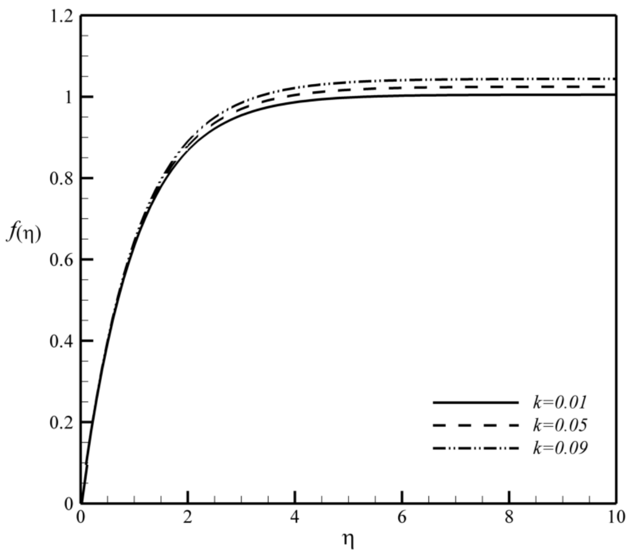

Fig. 4. vertical velocity profiles for several values of *k* with $\sigma = 1$

Fig. 5 shows the variation of several values of σ at a fixed value of elasticity, $k = 0.01$. It is observed that for a fixed value of k, temperature at a point decreases by increasing the Prandtl number, σ. The same trend can be seen for $k = 0.05$ and $k = 0.09$ in Fig. 6 and Fig. 7 respectively.

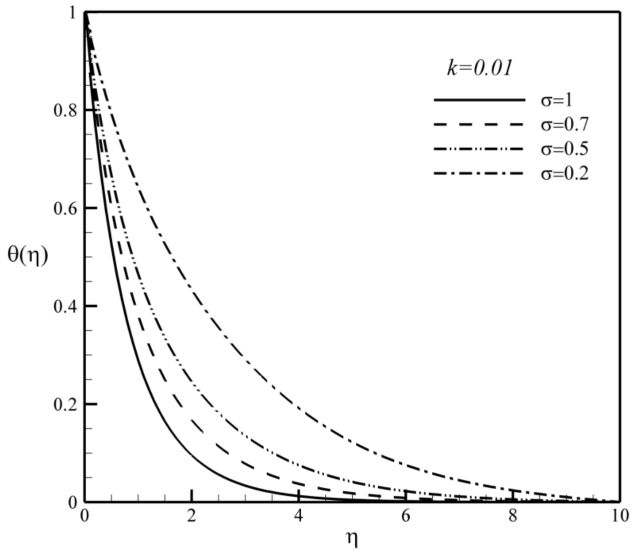

Fig. 5. temperature profiles for several values of σ with $k = 0.01$

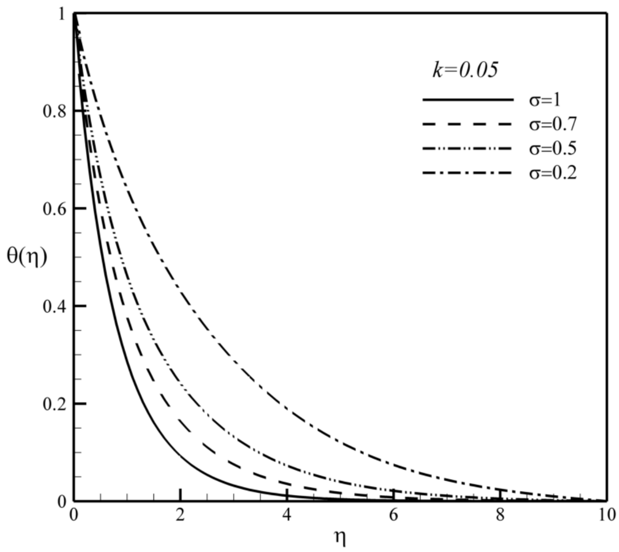

Fig. 6. temperature profiles for several values of σ with $k = 0.05$

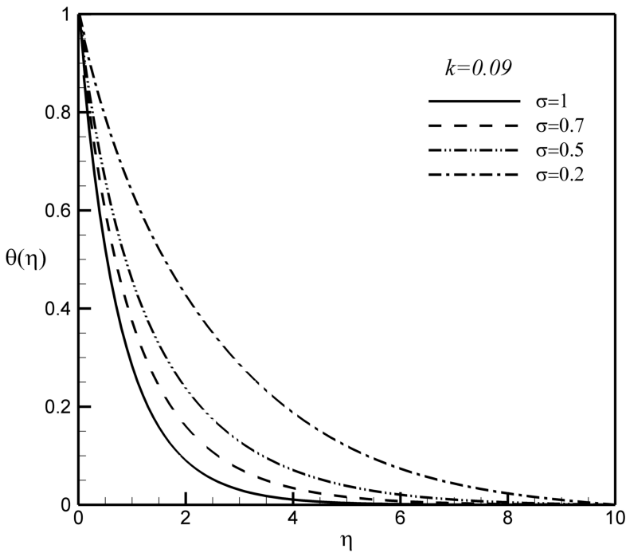

Fig. 7. temperature profiles for several values of σ with $k = 0.09$

It is observed that from Fig. 8 that for a fixed σ, temperature at a point decreases by increasing the elastic parameter k.

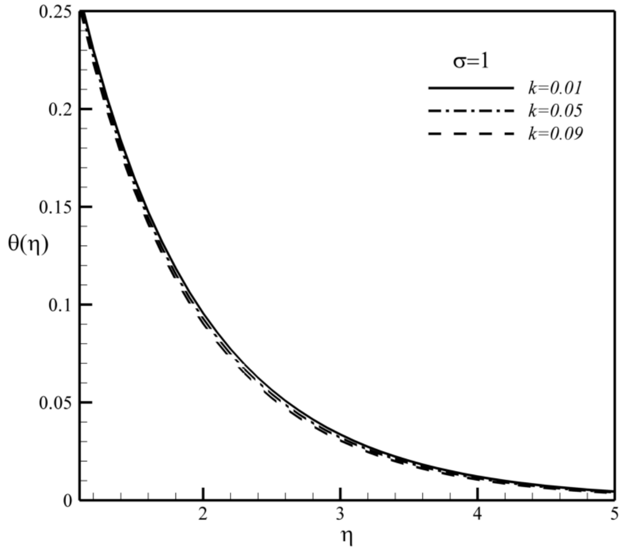

Fig. 8. temperature profiles for several values of k with $\sigma = 1$

Tables 1 and 2 are prepared for HAM results. These results are obtained for different values of k. For comparison the results obtained by HAM are compared by numerical solution by Cortell [36].

Table 1. The results of HAM and Numerical methods (NM) for $f(\eta)$ and $f'(\eta)$.

	η	f_{HAM}	f_{NM}	f'_{HAM}	f'_{NM}
$k = 0.01$	0	0	0	1	1
	0.1	0.0951858	0.0951858	0.905287	0.905287
	0.2	0.1813562	0.1813562	0.819544	0.819544
	0.5	0.3939173	0.3939174	0.608037	0.608038
	1	0.6334338	0.6334339	0.369709	0.36971
	2	0.86762	0.8676206	0.136685	0.136685
	5	0.9980414	0.9980459	0.006905	0.006907
	10	1.004924	1.0049396	4.5×10^{-5}	4.77×10^{-5}
$k = 0.05$	0	0	0	1	1
	0.1	0.095275	0.095275	0.907021	0.907021
	0.2	0.181692	0.181692	0.822686	0.822686
	0.5	0.613884	0.613883	0.395652	0.395651
	1	0.376853	0.376851	0.638536	0.638535
	2	0.142018	0.142015	0.87917	0.879166
	5	0.007601	0.007591	1.016907	1.016882
	10	1.024551	1.024636	4.5×10^{-5}	5.78×10^{-5}
$k = 0.09$	0	0	0	1	1
	0.1	0.09536	0.09536	0.908661	0.908662
	0.2	0.18201	0.18201	0.825665	0.825666
	0.5	0.397299	0.397299	0.619455	0.619456
	1	0.643407	0.643409	0.383723	0.383726
	2	0.879166	0.87917	0.142015	0.142018

5	1.035297	1.035345	0.008299	0.00832
10	1.043794	1.043958	4.54×10^{-5}	6.9×10^{-5}

Table 2. The results of HAM and Numerical methods (NM) for $\theta(\eta)$ and $\theta'(\eta)$ with $\sigma=1$.

	η	θ_{HAM}	θ_{NM}	$-\theta'_{HAM}$	$-\theta'_{NM}$
$k=0.01$	0	1	1	1.334725	1.334735
	0.1	0.875997	0.875997	1.150399	1.150410
	0.2	0.768992	0.768990	0.993962	0.993973
	0.5	0.526393	0.526390	0.650449	0.650461
	1	0.289529	0.289520	0.335672	0.335684
	2	0.095617	0.095600	0.10214	0.102150
	5	0.00442	0.004381	0.004441	0.004444
	10	4.54×10^{-5}	2×10^{-6}	2.9×10^{-5}	2.9×10^{-5}
$k=0.05$	0	1	1	1.34009	1.340102
	0.1	0.875466	0.875465	1.15562	1.155634
	0.2	0.767954	0.767953	0.99883	0.998838
	0.5	0.524128	0.524125	0.6537	0.653708
	1	0.286306	0.286300	0.33642	0.336430
	2	0.092879	0.092862	0.10101	0.101015
	5	0.004049	0.004013	0.00414	0.004146
	10	4.54×10^{-5}	4.2×10^{-5}	2.5×10^{-5}	2.5×10^{-5}
$k=0.09$	0	1	1	1.34515	1.345162
	0.1	0.874964	0.874964	1.16055	1.160560
	0.2	0.766975	0.766975	1.00342	1.003424
	0.5	0.521993	0.521991	0.65675	0.656761
	1	0.283277	0.283272	0.33709	0.337108
	2	0.090328	0.090314	0.099921	0.099930
	5	0.003719	0.003683	0.003871	0.003876
	10	4.54×10^{-5}	3.9×10^{-5}	2.1×10^{-5}	2.1×10^{-5}

6. Conclusion:

Explicit analytical solutions for the velocity field and rate of heat transfer are obtained for the viscoelastic fluid over a stretching sheet by using the Homotopy Analysis Method. This method provides highly accurate analytical solutions for nonlinear problems in comparison with other methods. The auxiliary parameter $\hbar$ provides us with a convenient way to adjust and control the convergence and its rate for the solutions series. These results are found to be in good agreement with numerical solutions obtained by [36].

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